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BRST cohomology operators on string superforms

Dao Vong Duc and Nguyen Thi Hong

Institute of Theoretical Physics, Academy of Sciences of Vietnam, Nghia Do, Tu Liem, Hanoi, Vietnam and International Centre for Theoretical Physics, Trieste, Italy

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Abstract. BRST cohomology calculus in the space of superstring differential forms is treated in detail. The Hodge star duality transformation is introduced and the explicit expressions of cohomology operators are derived for superforms of arbitrary order.

1. Introduction

Supersymmetric string theories [1-4] can be viewed as serious candidates for a consistent unified theory of all fundamental interactions. They exhibit more and more remarkable mathematical structures and link together different ideas and methods in elementary particle physics.

There exist several approaches to string theory; among these, especially promising is the recently developed version in which gauge symmetry and Lorentz covariance are displayed explicitly [5-10]. In all constructions of this type the BRST symmetry and the Fadeev-Popov ghosts play essential roles. In finding invariant equations for string functionals the formalism proves to be very convenient, based on cohomology calculus in the space of differential forms, considering the ghosts as differentials. This idea was initiated by Banks and Peskin [7] and has been further developed in a series of papers (for example [11-15]). In particular, Frenkel *et al* [14] have explained the similarity of the Banks-Peskin formalism with Kahler geometry and set up a general theory in terms of a 'semi-infinite' cohomology. Bars and Yankielowicz [15] have discussed in detail the interpretation of the formalism as a new kind of differential geometry.

The aim of this paper is to investigate further some issues along these lines. In particular, we introduce the Hodge star duality transformation for the bosonic string cohomology and extend the formalism to the Ramond and Neveu-Schwarz cases.

2. Bosonic string cohomology; Hodge star duality transformation

The string differential $\binom{p}{q}$ form is defined as [11, 13]:

$$\omega_{(q)}^{(p)} \equiv \omega_{n_1 \ldots n_q}^{m_1 \ldots m_p} e^{n_q} \ldots e^{n_1} e_{m_p} \ldots e_{m_1} \tag{2.1}$$

where $\{e_m\}$ and $\{e^n\}$ are the sets of anticommuting dual bases of $\binom{1}{0}$ and $\binom{0}{1}$ forms, m, $n = 1, 2, \ldots, \omega_{n_1 \ldots n_q}^{m_1 \ldots m_p}$ are functionals of string coordinates $x^\mu(\sigma)$ and can be considered to be antisymmetric with respect to upper and lower indices. The ordinary string field $\omega[x(\sigma)]$ is thought of as the $\binom{0}{0}$ form.

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The exterior derivative operator d is defined by the formula

$$\mathrm{d}\omega^{(p)}_{(q)} = (L_{n_{q+1}}\omega^{m_1\ldots m_p}_{n_1\ldots n_q} + \tfrac{1}{2}qV^{k}_{n_q n_{q+1}}\omega^{m_1\ldots m_p}_{n_1\ldots n_{q-1}k} + pW^{m_p}_{k n_{q+1}}\omega^{m_1\ldots m_{p-1}k}_{n_1\ldots n_q})$$
$$\times e^{n_{q+1}}e^{n_q}\ldots e^{n_1}e_{m_p}\ldots e_{m_1} \tag{2.2}$$

which can be obtained by putting

$$\begin{aligned} \mathrm{d}e_m &= W^{j}_{mk}e^{k}e_j \\ \mathrm{d}e^{n} &= \tfrac{1}{2}V^{n}_{km}e^{m}e^{k} \\ \mathrm{d}[\omega^{(p_1)}_{(q_1)}\cdot\omega^{(p_2)}_{(q_2)}] &= \mathrm{d}\omega^{(p_1)}_{(q_1)}\cdot\omega^{(p_2)}_{(q_2)} + (-1)^{p_1+q_1}\omega^{(p_1)}_{(q_1)}\cdot\mathrm{d}\omega^{(p_2)}_{(q_2)} \end{aligned} \tag{2.3}$$

where

$$V^{k}_{nm} = (n-m)\delta_{k,n+m} \qquad W^{k}_{nm} = (n+m)\delta_{k,n-m}. \tag{2.4}$$

L_n are the generators of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}Dn(n^2-1)\delta_{n+m,o} \tag{2.5}$$

D being the dimension of spacetime.

Let us define the operator $\bar{\mathrm{d}}$ which is different from d only by replacement of L_n by $-L_{-n}$ when acting on the $\binom{0}{0}$ form, namely

$$\bar{\mathrm{d}}\omega^{(p)}_{(q)} = [-L_{-n_{q+1}}\omega^{m_1\ldots m_p}_{n_1\ldots n_q} + \tfrac{1}{2}qV^{k}_{n_q n_{q+1}}\omega^{m_1\ldots m_p}_{n_1\ldots n_{q-1}k} + pW^{m_p}_{k n_{q+1}}\omega^{m_1\ldots m_{p-1}k}_{n_1\ldots n_q}]$$
$$\times e^{n_{q+1}}e^{n_q}\ldots e^{n_1}e_{m_p}\ldots e_{m1}. \tag{2.6}$$

Like d, $\bar{\mathrm{d}}$ is nilpotent.

Let Ω_N be the space of all $\binom{p}{q}$ forms with p, $q \leq N$ and the coefficients $\omega^{m_1\ldots m_p}_{n_1\ldots n_q}$ satisfying the condition

$$L_{-K_1}\ldots L_{-K_r}\omega^{m_1\ldots m_p}_{n_1\ldots n_q} = 0 \tag{2.7}$$

for arbitrary r, whenever at least one of the indices k, m, n takes a value greater than N. It is not difficult to show that this definition of Ω_N is invariant under the operator $\bar{\mathrm{d}}$, i.e. if $\omega^{(p)}_{(q)} \in \Omega_N$, then $\bar{\mathrm{d}}\omega^{(p)}_{(q)} \in \Omega_N$. We now define the duality transformation $*_{(N)}$ in the space Ω_N by the formula

$$\underset{(N)}{*}\ \omega^{(p)}_{(q)} = \frac{1}{(N-p)!(N-q)!}\varepsilon_{m_1\ldots m_N}\varepsilon^{n_1\ldots n_N}\omega^{m_1\ldots m_p}_{n_1\ldots n_q}e^{m_N}\ldots e^{m_{p+1}}e_{n_N}\ldots e_{n_{q+1}} \tag{2.8}$$

where $\varepsilon_{m_1\ldots m_N}(\varepsilon^{n_1\ldots n_N})$ is totally antisymmetric with respect to $m(n)$ with

$$\varepsilon_{12\ldots N} = \varepsilon^{12\ldots N} = 1.$$

It is clear from the definition (2.8) that under the operator $*_{(N)}$, a $\binom{p}{q}$ form $\in \Omega_N$ transforms into a $\binom{N-q}{N-p}$ form $\in \Omega_N$. In particular, we have:

$$\begin{aligned} &\underset{(N)}{*}\ 1 = e^{N}\ldots e^{1}e_N\ldots e_1 \\ &\underset{(N)}{*}\ e^{N}\ldots e^{1}e_N\ldots e_1 = 1 \\ &\underset{(N)}{*}\ \underset{(N)}{*}\ \omega^{(p)}_{(q)} = (-1)^{(p+q)(N+1)}\omega^{(p)}_{(q)}. \end{aligned} \tag{2.9}$$

Further, let us define the co-derivative operator δ in the following manner:

$$\begin{aligned} &\delta\omega^{(0)}_{(q)} = 0 \\ &\delta\omega^{(p)}_{(q)} = -(-1)^{(p+q)N}\underset{(N)}{*}\ \bar{\mathrm{d}}\ \underset{(N)}{*}\ \omega^{(p)}_{(q)} \qquad p>0. \end{aligned} \tag{2.10}$$

The direct calculations give:

$$\delta\omega^{(p)}_{(q)} = (-1)^q p[L_{-K}\omega^{m_1\ldots m_{p-1}K}_{n_1\ldots n_q} - \tfrac{1}{2}(p-1)V^{m_{p-1}}_{kl}\omega^{m_1\ldots m_{p-2}lk}_{n_1\ldots n_q}$$
$$+ qW^l_{n_q k}\omega^{m_1\ldots m_{p-1}K}_{n_1\ldots n_{q-1}l}]e^{n_q}\ldots e^{n_1}e_{m_{p-1}}\ldots e_{m_1}. \tag{2.11}$$

So, the operator δ transforms a $\binom{p}{q}$ form into a $\binom{p-1}{q}$ form and its explicit expression does not depend on N.

The nilpotency of δ follows from the definition (2.10), equations (2.9) and the nilpotency of $\bar{\mathrm{d}}$.

By defining the inner product of two forms $\alpha^{(p)}_{(q)}$ and $\beta^{(q)}_{(p)}$ as

$$\langle\alpha^{(p)}_{(q)}\cdot\beta^{(q)}_{(p)}\rangle \equiv (\alpha^{m_1\ldots m_p}_{n_1\ldots n_q}|\beta^{n_1\ldots n_q}_{m_1\ldots m_p}) \tag{2.12}$$

with the property

$$(\alpha^{m_1\ldots m_p k}_{n_1\ldots n_q}|L_k\beta^{n_1\ldots n_q}_{m_1\ldots m_p}) = (L_{-k}\alpha^{m_1\ldots m_p k}_{n_1\ldots n_q}|\beta^{n_1\ldots n_q}_{m_1\ldots m_p}) \tag{2.13}$$

we have the duality relation:

$$\langle\delta\alpha^{(p)}_{(q)}\cdot\beta^{(q)}_{(p-1)}\rangle = (-1)^q\langle\alpha^{(p)}_{(q)}\cdot \mathrm{d}\beta^{(q)}_{(p-1)}\rangle. \tag{2.14}$$

3. Superstring cohomology

In the superstring case the differential superform $\binom{p}{q}$ is defined as the generalisation of (2.1):

$$\omega^{(p)}_{(q)} \equiv \omega^{A_1\ldots A_p}_{B_1\ldots B_q}e^{B_q}\ldots e^{B_1}e_{A_p}\ldots e_{A_1}. \tag{3.1}$$

Here the superindices $A \equiv n, \lambda$ are introduced with λ taking positive half-integer values for the Neveu–Schwarz sector and positive integer for the Ramond sector. The dual basis forms $\{e_A\}$ and $\{e^B\}$ satisfy the permutation law

$$e_{A_1}e_{A_2} = -(A_1)(A_2)(A_1, A_2)e_{A_2}e_{A_1} \tag{3.2}$$

and analogously for $e^{B_1}e^{B_2}$, $e_A e^B$. We have used the notation

$$(A) \equiv (-1)^{[A]}$$
$$(A_1A_2\ldots, B_1B_2\ldots) \equiv (-1)^{([A_1]+[A_2]+\ldots)([B_1]+[B_2]+\ldots)} \tag{3.3}$$

with $[A]$ being the grading of index A, namely $[n]=0$, $[\lambda]=1$.

In accordance with (3.2) we can always consider $\omega^{A_1\ldots A_p}_{B_1\ldots B_q}$ to have the similar symmetry property with respect to upper and lower indices.

The exterior derivative operator d is defined by its action on the $\binom{0}{0}$ superform and on e_A, e^B, which is the generalisation of (2.3):

$$\mathrm{d}\omega = F_B\omega e^B$$
$$\mathrm{d}e_A = (B)W^C_{AB}e^B e_C \tag{3.4}$$
$$\mathrm{d}e^B = (D)\tfrac{1}{2}V^B_{CD}e^D e_C$$

with Leibnitz rule

$$\mathrm{d}[e_A\omega^{(p)}_{(q)}] = \mathrm{d}e_A\,\omega^{(p)}_{(q)} - (A)e_A\,\mathrm{d}\omega^{(p)}_{(q)}. \tag{3.5}$$

Here $F_A \equiv L_n$, G_λ stand for the generators of the super-Virasoro algebra

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{8}D_m(m^2 - r)\delta_{m+n,0} \\
&[L_m, G_\lambda] = (\tfrac{1}{2}m - \lambda)G_{m+\lambda} \\
&\{G_\lambda, G_\sigma\} = 2L_{\lambda+\sigma} + \tfrac{1}{8}D(4\lambda^2 - r).
\end{aligned}
\tag{3.6}
$$

$r = 1$ for Neveu-Schwarz sector and 0 for Ramond sector. V^C_{AB} and W^C_{AB} denote the structure constants (of these the non-vanishing values are given in (2.4)) and

$$
V^p_{\lambda\sigma} = 2\delta_{\beta,\lambda+\sigma} \qquad V^\tau_{n\lambda} = -V^\tau_{\lambda n} = \left(\frac{n}{2} - \lambda\right)\delta_{\tau,n+\lambda}
\tag{3.7}
$$

$$
W^p_{\lambda\sigma} = 2\delta_{p,\lambda-\sigma} \qquad W^\tau_{n\lambda} = \left(\frac{n}{2} + \lambda\right)\delta_{\tau,n-\lambda} \qquad W^\tau_{\lambda n} = \left(\frac{n}{2} + \lambda\right)\delta_{\tau,\lambda-n}.
$$

The calculations give the following result:

$$
\begin{aligned}
\mathrm{d}\omega^{(p)}_{(q)} = [&F_{B_{q+1}}\omega^{A_1 \ldots A_p}_{B_1 \ldots B_q} + (B_{qH})\tfrac{1}{2}qV^C_{B_q B_{q+1}}\omega^{A_1 \ldots A_p}_{B_1 \ldots B_{q-1}C} \\
&+ (B_{q+1})^{q+1}(B_{q+1}, B_1 \ldots B_q)pW^{A_p}_{CB_{q+1}}\omega^{A_1 \ldots A_{p-1}C}_{B_1 \ldots B_q}] \\
&\times e^{B_{q-1}}e^{B_q} \ldots e^{B_1}e_{A_r} \ldots e_{A_1}
\end{aligned}
\tag{3.8}
$$

which is the generalisation of (2.2).

The nilpotency of d can be verified, using the following identities for the structure constants:

$$
\begin{aligned}
&V^D_{AB}V^E_{CD} + (AB, C)V^D_{BC}V^E_{AD} + (CA, B)V^D_{CA}V^E_{BD} = 0 \\
&V^D_{CA}W^E_{BD} - W^D_{BA}W^E_{DC} + (A, C)W^D_{BC}W^E_{DA} = 0.
\end{aligned}
\tag{3.9}
$$

For the co-derivative operator δ we find the following generalisation of (2.11):

$$
\begin{aligned}
\delta\omega^{(p)}_{(q)} = (-1)^q p(B_1) \ldots (B_q)(C, B_1 \ldots B_q)(C)^q \\
\times [F_{-C}\omega^{A_1 \ldots A_{p-1}C}_{B_1 \ldots B_q} - \tfrac{1}{2}(p-1)(C, A_{p-1}B_1 \ldots B_q)(C)^q V^{A_{p-1}}_{CD}\omega^{A_1 \ldots A_{p-2}DC}_{B_1 \ldots B_q} \\
+ (C)qW^D_{B_q C}\omega^{A_1 \ldots A_{p-1}C}_{B_1 \ldots B_{q-1}D}]e^{B_q} \ldots e^{B_1}e_{A_{p-1}} \ldots e_{A_1}.
\end{aligned}
\tag{3.10}
$$

Finally, let us quote briefly the isomorphism between the formalism based on cohomology in the space of differential forms and the BRST formalism.

Denote the superconformal ghosts and antighosts related to the coordinate reparametrisation by g_A and $\bar{g}_A$. They satisfy the commutation relations:

$$
\begin{aligned}
&[g_A, \bar{g}_B]_{(A)(B)(A,B)} \equiv g_A\bar{g}_B + (A)(B)(A, B)\bar{g}_B g_A = \delta_{A+B,0} \\
&[g_A, g_B]_{(A)(B)(A,B)} = [\bar{g}_A, \bar{g}_B]_{(A)(B)(A,B)} = 0
\end{aligned}
\tag{3.11}
$$

and the Hermiticity condition:

$$
g^+_A = g_{-A} \qquad \bar{g}^+_A = (A)\bar{g}_{-A}.
$$

Consider the state $\omega^{(p)}_{(q)}$ constructed in Fock space on the vacuum $|0\rangle$ in the following manner:

$$
\omega^{(p)}_{(q)} \equiv \sum_{A,B>0} \omega^{A_1 \ldots A_p}_{B_1 \ldots B_q} g_{-B_q} \ldots g_{-B_1}\bar{g}_{-A_p} \ldots \bar{g}_{-A_1}|0\rangle
\tag{3.12}
$$

with the vacuum satisfying the condition

$$
g_A|0\rangle = \bar{g}_A|0\rangle = 0 \qquad A > 0.
$$

By defining

$$\mathrm{d} \equiv \sum_{A>0} F_A g_{-A} - \sum_{A,B,C>0} (B, C)[\tfrac{1}{2} V^C_{AB} g_{-A} g_{-B} \bar{g}_C + W^C_{BA} \bar{g}_{-C} g_{-A} g_B] \tag{3.13}$$

we find that the actiun of d and $\mathrm{d}^\dagger$ on the state $\omega^{(p)}_{(q)}$ defined in (3.11) is given by formulae quite analogous to (3.8) and (3.10). So, we have the following isomorphism:

$$e^{B_q} \dots e^{B_1} e_{A_p} \dots e_{A_1} \leftrightarrow g_{-B_q} \dots g_{-B_1} \bar{g}_{-A_p} \dots \bar{g}_{-A_1} |0\rangle \qquad \mathrm{d} \leftrightarrow \mathrm{d} \qquad \delta \leftrightarrow \mathrm{d}^\dagger.$$

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